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Evaluating the Fractional Integrals of Some Fractional Rational Functions

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Abstract: **In this paper, the fractional integration problem of fractional rational functions is studied based on Jumarie's modified Riemann-Liouville (R-L) fractional calculus and a new multiplication of fractional analytic functions. The main methods we used are the chain rule for fractional derivatives and the partial fraction method. On the other hand, we give some examples to illustrate how to calculate fractional integrals of some fractional rational functions. In fact, these results are extensions of the results in traditional calculus.**

Keywords: **Fractional rational functions, Jumarie's modified R-L fractional calculus, New multiplication, Fractional analytic functions, Chain rule, Partial fraction method.**

I. INTRODUCTION

Fractional calculus is the theory of non-integer derivative and integral. However, the definition of fractional derivative is not unique. Common definitions include Riemann Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald Letnikov (G-L) fractional derivative and Jumarie's modification of R-L fractional derivative [1-5]. In the past decades, fractional calculus has been widely used in continuum mechanics, quantum mechanics, electronic engineering, fluid science, viscoelasticity, control theory, dynamics, financial economics and other fields [6-14].

In this paper, based on the Jumarie type of modified R-L fractional calculus and a new multiplication of fractional analytic functions, the fractional integration problem of fractional rational functions is studied. The major methods used in this article are the chain rule for fractional derivatives and the partial fraction method. In addition, some examples are provided to illustrate our methods. In fact, these results we obtained are generalizations of those in classical calculus.

II. DEFINITIONS AND PROPERTIES

Firstly, the fractional calculus used in this paper is introduced.

Definition 2.1 ([15])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. The Jumarie type of Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\,. \tag{1}$$

And the Jumarie′s modified R-L α-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\,, \tag{2}$$

where $\Gamma(\ \ )$ is the gamma function.

Proposition 2.2 ([16])**:** *If* α, β, x_0, C *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{3}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C] = 0. \tag{4}$$

In the following, we introduce the definition of fractional analytic function.

Definition 2.3 ([17])**:** Suppose that x, x_0, and a_k are real numbers for all k, $x_0 \in (a,b)$, and $0<\alpha\leq 1$. If the function $f_\alpha:[a,b]\to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. In addition, if $f_\alpha:[a,b]\to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, a new multiplication of fractional analytic functions is introduced.

Definition 2.4 ([15])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. If $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}, \tag{5}$$

$$g_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}=\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}. \tag{6}$$

Then we define

$$\begin{aligned}&f_\alpha(x^\alpha)\otimes g_\alpha(x^\alpha)\\&=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}\otimes\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}\\&=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)(x-x_0)^{k\alpha}.\end{aligned} \tag{7}$$

Equivalently,

$$\begin{aligned}&f_\alpha(x^\alpha)\otimes g_\alpha(x^\alpha)\\&=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}\otimes\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}\\&=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}.\end{aligned} \tag{8}$$

Definition 2.5: Let $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ be two α-fractional analytic functions defined on an interval containing x_0. If $f_\alpha(x^\alpha)\otimes g_\alpha(x^\alpha)=1$, then we say that $g_\alpha(x^\alpha)$ is the $\otimes$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $[f_\alpha(x^\alpha)]^{\otimes -1}$.

Definition 2.6 ([18])**:** Suppose that $0<\alpha\leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}, \tag{9}$$

$$g_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}=\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}. \tag{10}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha\circ g_\alpha)(x^\alpha)=f_\alpha(g_\alpha(x^\alpha))=\sum_{k=0}^{\infty}\frac{a_k}{k!}(g_\alpha(x^\alpha))^{\otimes k}, \tag{11}$$

and

$$(g_\alpha\circ f_\alpha)(x^\alpha)=g_\alpha(f_\alpha(x^\alpha))=\sum_{k=0}^{\infty}\frac{b_k}{k!}(f_\alpha(x^\alpha))^{\otimes k}. \tag{12}$$

Definition 2.7 ([18])**:** Let $0<\alpha\leq 1$. If $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions satisfies

$$(f_\alpha\circ g_\alpha)(x^\alpha)=(g_\alpha\circ f_\alpha)(x^\alpha)=\frac{1}{\Gamma(\alpha+1)}x^{\alpha}. \tag{13}$$

Then $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are called inverse functions of each other.

Some fractional analytic functions are introduced below.

Definition 2.8 ([18])**:** If $0 < \alpha \leq 1$, and x, x_0 are real numbers. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}. \tag{14}$$

And the α-fractional logarithmic function $Ln_\alpha(x^\alpha)$ is the inverse function of $E_\alpha(x^\alpha)$.

Next, we introduce fractional rational function.

Definition 2.9: Let $0 < \alpha \leq 1$, and $c_m \neq 0$. Then

$$P_\alpha(x^\alpha) = \sum_{k=0}^{m} c_k \cdot \frac{x^{k\alpha}}{\Gamma(k\alpha+1)} = \sum_{k=0}^{m} \frac{c_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}. \tag{15}$$

is called an α-fractional polynomial function of degree m. Furthermore, if $P_\alpha(x^\alpha)$ and $Q_\alpha(x^\alpha)$ are two α-fractional polynomial functions, $Q_\alpha(x^\alpha) \neq 0$, then $R_\alpha(x^\alpha) = P_\alpha(x^\alpha) \otimes [Q_\alpha(x^\alpha)]^{\otimes -1}$ is called an α-fractional rational function.

Theorem 2.10 (chain rule for fractional derivatives) ([18])**:** *If* $0 < \alpha \leq 1$, x, x_0 *are real numbers, and* $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ *are* α*-fractional analytic functions at* x_0 *. Then*

$$\left({}_{x_0}D_x^\alpha\right)\left[f_\alpha\left(g_\alpha(x^\alpha)\right)\right] = \left({}_{x_0}D_x^\alpha\right)[f_\alpha(x^\alpha)]\left(g_\alpha(x^\alpha)\right) \otimes \left({}_{x_0}D_x^\alpha\right)[g_\alpha(x^\alpha)]. \tag{16}$$

III. RESULTS AND EXAMPLES

In the following, we obtain the main results in this article.

Theorem 3.1: *Let* p, α, x *be real numbers,* $p > 0, x \geq 0,$ *and* $0 < \alpha \leq 1$. *Then the* α*-fractional intregral*

$$\left({}_0I_x^\alpha\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha + p\right)^{\otimes -1}\right] = Ln_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha + p\right) - Ln_\alpha(p)\,. \tag{17}$$

Proof Since by chain rule for fractional derivatives,

$$\left({}_0D_x^\alpha\right)\left[Ln_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha + p\right)\right] = \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha + p\right)^{\otimes -1}. \tag{18}$$

It follows that the desired result holds. Q.e.d.

Theorem 3.2: *Suppose that* p, α, x *are real numbers,* n *is a positive integer,* $n \geq 2$, $p > 0, x \geq 0,$ *and* $0 < \alpha \leq 1$. *Then the* α*-fractional intregral*

$$\left({}_0I_x^\alpha\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha + p\right)^{\otimes -n}\right] = \frac{1}{-n+1}\cdot\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha + p\right)^{\otimes(-n+1)} - \frac{p^{(-n+1)}}{-n+1}. \tag{19}$$

Proof Using chain rule for fractional derivatives yields

$$\left({}_0D_x^\alpha\right)\left[\frac{1}{-n+1}\cdot\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha + p\right)^{\otimes(-n+1)}\right] = \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha + p\right)^{\otimes -n}. \tag{20}$$

Therefore, the desired result holds. Q.e.d.

Next, we give two examples to illustrate how to use Theorems 3.1 and 3.2 and the partial fraction method to calculate the fractional integrals of some fractional rational functions.

Example 3.3: Let $x \geq 0$, and $0 < \alpha \leq 1$. Find the α-fractional integral

$$\left({}_0I_x^\alpha\right)\left[\left(7\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha + 23\right) \otimes \left(\frac{2}{\Gamma(2\alpha+1)}x^{2\alpha} + 7\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha + 10\right)^{\otimes -1}\right]. \tag{21}$$

Solution By partial fraction method, the α-fractional rational function

$$\begin{aligned}
&\left(7\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha + 23\right) \otimes \left(\frac{2}{\Gamma(2\alpha+1)}x^{2\alpha} + 7\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha + 10\right)^{\otimes -1} \\
&= \left[4\cdot\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha + 2\right) + 3\cdot\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha + 5\right)\right] \otimes \left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha + 2\right) \otimes \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha + 5\right)\right]^{\otimes -1} \\
&= 4\cdot\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha + 5\right)^{\otimes -1} + 3\cdot\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha + 2\right)^{\otimes -1}.
\end{aligned} \tag{22}$$

It follows from Theorem 3.1 that

$$
\begin{aligned}
&\left({}_0I_x^{\alpha}\right)\left[\left(7\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+23\right)\otimes\left(\frac{2}{\Gamma(2\alpha+1)}x^{2\alpha}+7\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+10\right)^{\otimes-1}\right]\\
&=\left({}_0I_x^{\alpha}\right)\left[4\cdot\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+5\right)^{\otimes-1}+3\cdot\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+2\right)^{\otimes-1}\right]\\
&=4\cdot\left({}_0I_x^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+5\right)^{\otimes-1}\right]+3\cdot\left({}_0I_x^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+2\right)^{\otimes-1}\right]\\
&=4\cdot\left[Ln_{\alpha}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+5\right)-Ln_{\alpha}(5)\right]+3\cdot\left[Ln_{\alpha}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+2\right)-Ln_{\alpha}(2)\right]\\
&=4\cdot Ln_{\alpha}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+5\right)+3\cdot Ln_{\alpha}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+2\right)-4\cdot Ln_{\alpha}(5)-3\cdot Ln_{\alpha}(2). \quad (23)
\end{aligned}
$$

Example 3.4: If $x\geq 0$, and $0<\alpha\leq 1$. Find the α-fractional integral

$$
\left({}_0I_x^{\alpha}\right)\left[\left(32\cdot\frac{1}{\Gamma(2\alpha+1)}x^{2\alpha}+10\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+6\right)\otimes\left(2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right)^{\otimes-3}\right]. \quad (24)
$$

Solution Using partial fraction method yields the α-fractional rational function

$$
\begin{aligned}
&\left(32\cdot\frac{1}{\Gamma(2\alpha+1)}x^{2\alpha}+10\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+6\right)\otimes\left(2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right)^{\otimes-3}\\
&=\left[4\cdot\left(2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right)^{\otimes 2}-3\cdot\left(2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right)+5\right]\otimes\left(2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right)^{\otimes-3}\\
&=4\cdot\left(2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right)^{\otimes-1}-3\cdot\left(2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right)^{\otimes-2}+5\cdot\left(2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right)^{\otimes-3}. \quad (25)
\end{aligned}
$$

It follows from Theorems 3.1 and 3.2 that

$$
\begin{aligned}
&\left({}_0I_x^{\alpha}\right)\left[\left(32\cdot\frac{1}{\Gamma(2\alpha+1)}x^{2\alpha}+10\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+6\right)\otimes\left(2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right)^{\otimes-3}\right]\\
&=\left({}_0I_x^{\alpha}\right)\left[4\cdot\left(2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right)^{\otimes-1}-3\cdot\left(2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right)^{\otimes-2}+5\cdot\left(2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right)^{\otimes-3}\right]\\
&=4\cdot\left({}_0I_x^{\alpha}\right)\left[\left(2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right)^{\otimes-1}\right]-3\cdot\left({}_0I_x^{\alpha}\right)\left[\left(2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right)^{\otimes-2}\right]+5\cdot\left({}_0I_x^{\alpha}\right)\left[\left(2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right)^{\otimes-3}\right]\\
&=2\cdot\left({}_0I_x^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+\frac{1}{2}\right)^{\otimes-1}\right]-\frac{3}{4}\cdot\left({}_0I_x^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+\frac{1}{2}\right)^{\otimes-2}\right]+\frac{5}{8}\cdot\left({}_0I_x^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+\frac{1}{2}\right)^{\otimes-3}\right]\\
&=2\cdot\left[Ln_{\alpha}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+\frac{1}{2}\right)-Ln_{\alpha}\left(\frac{1}{2}\right)\right]-\frac{3}{4}\cdot\left[-\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+\frac{1}{2}\right)^{\otimes-1}+2\right]+\frac{5}{8}\left[\frac{1}{-2}\cdot\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+\frac{1}{2}\right)^{\otimes-2}+2\right]\\
&=2\cdot Ln_{\alpha}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+\frac{1}{2}\right)+\frac{3}{4}\cdot\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+\frac{1}{2}\right)^{\otimes-1}-\frac{5}{16}\cdot\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+\frac{1}{2}\right)^{\otimes-2}-2\cdot Ln_{\alpha}\left(\frac{1}{2}\right)-\frac{1}{4}. \quad (26)
\end{aligned}
$$

IV. CONCLUSION

Based on Jumarie's modified R-L fractional calculus and a new multiplication, the fractional integral of fractional rational functions is studied. We use the chain rule for fractional derivatives and the partial fraction method to calculate some fractional integrals. In fact, these results are the generalizations of classical calculus results. In the future, we will also use these methods to expand our research field to engineering mathematics and fractional differential equations.

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